Computation and Modeling Assignment 20

Anton Perez

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Problem 20-1

Consider the exponential distribution defined by

$$p_2(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

1. Using integration, show that this is a valid distribution, i.e. all the probability integrates to 1.

   Solution:

$$\begin{aligned} \int_{-\infty}^{\infty} p_2(x)\mathrm{d}x &= \int_{-\infty}^{0} 0\mathrm{d}x + \int_{0}^{\infty} 2e^{-2x}\mathrm{d}x \\ &= \int_{0}^{\infty} 2e^{-2x}\mathrm{d}x \\ &= -e^{-2x}\big|_0^\infty \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

2. Given that $X \sim p_2$, compute $P(0 < X \leq 1)$.

   Solution:

$$\begin{aligned} \int_{0}^{1} p_2(x)\mathrm{d}x &= \int_{0}^{1} 2e^{-2x}\mathrm{d}x \\ &= -e^{-2x}\big|_0^1 \\ &= -e^{-2} - (-1) \\ &= 1 - e^{-2} \end{aligned}$$

3. Given that $X \sim p_2$, compute $E[X]$.

   Solution:

$$\begin{aligned} \int_{-\infty}^{\infty} xp_2(x)\mathrm{d}x &= \int_{-\infty}^{0} 0x\mathrm{d}x + \int_{0}^{\infty} 2xe^{-2x}\mathrm{d}x \\ &= \int_{0}^{\infty} 2xe^{-2x}\mathrm{d}x \\ &= -xe^{-2x} - \frac{e^{-2x}}{2}\big|_0^\infty \\ &= 0 - (-\frac{1}{2}) \\ &= \frac{1}{2} \end{aligned}$$

4. Given that $X \sim p_2$, compute $\text{Var}[X]$.

Solution:

$$\begin{aligned}
\int_{-\infty}^{\infty} p_2(x)(x - E[X])^2 \mathrm{d}x &= \int_{-\infty}^{0} 0(x - E[X])^2 \mathrm{d}x + \int_{0}^{\infty} 2e^{-2x}(x - E[X])^2 \mathrm{d}x \\
&= \int_{0}^{\infty} 2e^{-2x}(x - \frac{1}{2})^2 \mathrm{d}x \\
&= -e^{-2x}(x - \frac{1}{2})^2 - e^{-2x}(x - \frac{1}{2}) - \frac{e^{-2x}}{2}\Big|_0^\infty \\
&= -(x^2 + \frac{1}{4})e^{-2x}\Big|_0^\infty \\
&= 0 - (-\frac{1}{4}) \\
&= \frac{1}{4}
\end{aligned}$$