Computation and Modeling Assignment 33

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Problem 33-1

Suppose you are given the following dataset:

$$\text{data} = [(1, 0.2), (2, 0.25), (3, 0.5)]$$

Fit a logistic regression model $y = \dfrac{1}{1 + e^{ax+b}}$

Solution:

$$ax + b = \ln\left(\frac{1}{y} - 1\right)$$

$$\begin{aligned} a + b &= \ln(4) = 1.386294 \\ 2a + b &= \ln(3) = 1.098612 \\ 3a + b &= \ln(1) = 0 \end{aligned}$$

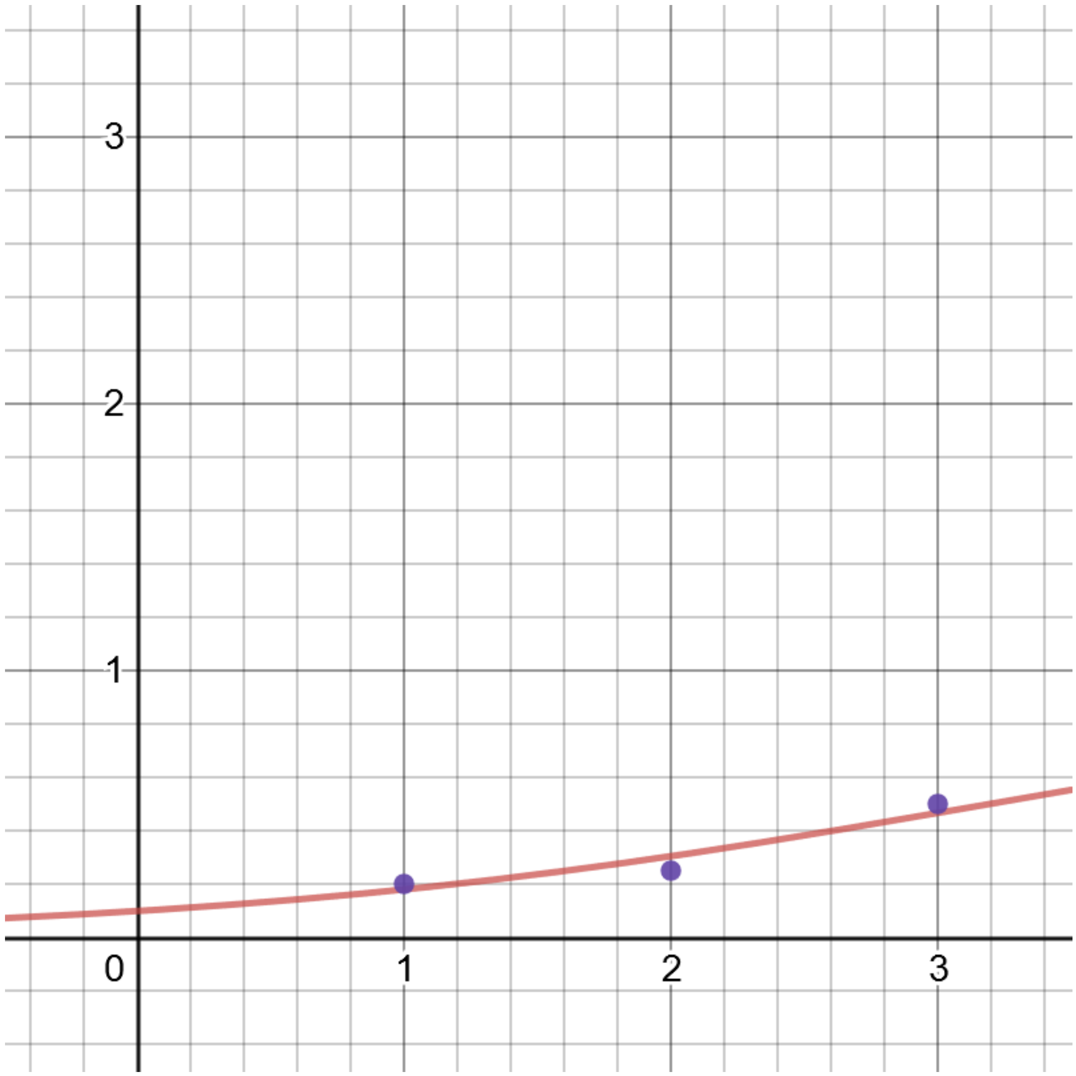
$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \ln(4) \\ \ln(3) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \ln(4) \\ \ln(3) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 14 & 6 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \ln(4) + 2\ln(3) \\ \ln(4) + \ln(3) \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -1 \\ -1 & \frac{7}{3} \end{bmatrix} \begin{bmatrix} \ln(36) \\ \ln(12) \end{bmatrix} = \begin{bmatrix} -0.693147 \\ 2.214597 \end{bmatrix}$$

$$y = \frac{1}{1 + e^{-0.693147x + 2.214597}}$$



Problem 33-2

1. Given that $X \sim p(x)$, where $p(x)$ is a continuous distribution, prove that for any real number a we have $E[aX] = aE[X]$.

   Solution:

$$\begin{aligned} E[aX] &= \int_{-\infty}^{\infty} (ax)p(x) \, \mathrm{d}x \\ &= a \int_{-\infty}^{\infty} xp(x) \, \mathrm{d}x \\ &= aE[X] \end{aligned}$$

2. Given that $X \sim p(x)$ where p(x) is a continuous probability distribution, prove the identity $\text{Var}[X] = E[X^2] - E[X]^2$.

Solution:

$$\begin{aligned}
\mathrm{Var}[X] &= \int_{-\infty}^{\infty} (x - E[X])^2 p(x) \ \mathrm{d}x \\
&= \int_{-\infty}^{\infty} x^2 p(x) - 2E[X]xp(x) + E[X]^2 p(x) \ \mathrm{d}x \\
&= \int_{-\infty}^{\infty} x^2 p(x) \ \mathrm{d}x - 2E[X] \int_{-\infty}^{\infty} xp(x) \ \mathrm{d}x + E[X]^2 \int_{-\infty}^{\infty} p(x) \ \mathrm{d}x \\
&= E[X^2] - 2E[X]^2 + E[X]^2 \\
&= E[X^2] - E[X]^2
\end{aligned}$$