Machine Learning Assignment 49

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49-1

(A) $E[a \cdot X] = a \cdot E[X]$

$$\begin{aligned} E[a \cdot X] &= \int_{\infty}^{\infty} (a \cdot x) \cdot p(x)\, dx \\ &= a \cdot \int_{\infty}^{\infty} x \cdot p(x)\, dx \\ &= a \cdot E[X] \end{aligned}$$

(B) $E[X_1 + X_2] = E[X_1] + E[X_2]$

$$\begin{aligned} E[X_1 + X_2] &= \int_{\infty}^{\infty} (X_1 + X_2) \cdot p(x)\, dx \\ &= \int_{\infty}^{\infty} X_1 \cdot p(x) + X_2 \cdot p(x)\, dx \\ &= \int_{\infty}^{\infty} X_1 \cdot p(x) + \int_{\infty}^{\infty} X_2 \cdot p(x)\, dx \\ &= E[X_1] + E[X_2] \end{aligned}$$

(C) $Var[X] = E[X^2] - E[X]^2$

$$\begin{aligned} Var[X] &= E[(x - E[x])^2] \\ &= E[(X - E[X])2] \\ &= E[X^2 - 2 \cdot X \cdot E[X] + E[X]^2] \\ &= E[X^2] - 2 \cdot E[X] \cdot E[X] + (E[X])^2 \\ &= E[X^2] - E[X]^2 \end{aligned}$$

(D) Bisection Search of $\sqrt{5}$ to 4 decimal places.

Because $\sqrt{4} = 2$ and $\sqrt{9} = 3$ and since $4 < 5 < 9$ the initial bounds are [2,3], and the to find the $\sqrt{5}$ we have to make a function such that $f(\sqrt{5}) = 0$ and the function that fits this is $f(x) = x^2 - 5$, so we compute,

Bounds	Mid-Point	F(Mid-Point)	Estimation
$[2, 2.5]$	2.5	1.25	2.25
$[2, 2.25]$	2.25	0.0625	2.125
$[2.125, 2.25]$	2.125	-0.484375	2.1875
$[2.1875, 2.25]$	2.1875	-0.21484375	2.21875
$[2.21875, 2.25]$	2.21875	-0.0771484375	2.234375
$[2.234375, 2.25]$	2.234375	-0.007568359375	2.2421875
$[2.234375, 2.2421875]$	2.2421875	0.02740478515625	2.23828125
$[2.234375, 2.23828125]$	2.23828125	0.0099029541015625	2.236328125
$[2.234375, 2.236328125]$	2.236328125	0.001163482666015625	2.2353515625
$[2.2353515625, 2.236328125]$	2.2353515625	-0.0032033920288085938	2.23583984375
$[2.23583984375, 2.236328125]$	2.23583984375	-0.001020193099975586	2.236083984375

We know the $\sqrt{5} = 2.23606798$ Using the final bounds the estimation 2.2360... and the first 4 decimal places of my Bisection search are the same as what we want. therefore I am done.

(E) Merge sort of $[4, 8, 7, 7, 4, 2, 3, 1]$

Input list:$[4, 8, 7, 7, 4, 2, 3, 1]$
Break it in half: $[4, 8, 7, 7]$, $[4, 2, 3, 1]$
Use Merge Sort recursively to sort the two halves
    Input list: $[4, 8, 7, 7]$
    Break it in half: $[4, 8]$, $[7, 7]$
    Use Merge Sort recursively to sort the two halves
        Input list: $[4, 8]$
        Break it in half: $[4]$, $[8]$
        The two halves have only one element each, so they are already sorted
        So we can combine them to get $[4, 8]$
        Input list: $[7, 7]$
        Break it in half: $[7]$, $[7]$
        The two halves have only one element each, so they are already sorted
        So we can combine them to get $[7, 7]$
    Now we have two sorted lists $[4, 8]$ and $[7, 7]$
    So we can combine them to get $[4, 7, 7, 8]$
    Input list: $[4, 2, 3, 1]$
    Break it in half: $[4, 2]$, $[3, 1]$
    Use Merge Sort recursively to sort the two halves
        Input list: $[4, 2]$
        Break it in half: $[4]$, $[2]$
        The two halves have only one element each, so they are already sorted
        So we can combine them to get $[2, 4]$
        Input list: $[3, 1]$
        Break it in half: $[3]$, $[1]$
        The two halves have only one element each, so they are already sorted
        So we can combine them to get $[1, 3]$
    Now we have two sorted lists $[2, 4]$ and $[1, 3]$
    So we can combine them to get $[1, 2, 3, 4]$
Now we have two sorted lists $[4, 7, 7, 8]$ and $[1, 2, 3, 4]$
So we can combine them to get $[1, 2, 3, 4, 4, 7, 7, 8]$

49-2

Part 1 Bayesian inference with a uniform prior

(A) $\mathcal{L}(k|1000 \text{ successes})$

$$\mathcal{L}(k|1000 \text{ successes}) = k \cdot k \cdot k ... 996 \text{ times} ... \cdot k \qquad = k^{1000}$$

(B) Normalized $\mathcal{L}(k|1000 \text{ successes})$

$$\begin{aligned}
\mathcal{L}(k|1000 \text{ successes}) &= \int_{-\infty}^{\infty} c * \mathcal{L}(k|1000 \text{ successes})\, dk \\
&= \int_{-\infty}^{\infty} c \cdot \mathcal{L}(k|1000 \text{ successes})\, dk \\
&= \int_{-\infty}^{\infty} c \cdot k^{1000}\, dk \\
&= \int_{0}^{1} c \cdot k^{1000}\, dk \\
&= c \cdot (\frac{(1)^1 001}{1001} - \frac{(0)^1 001}{1001}) \\
&\implies c \cdot \frac{1}{1001} = 1 \\
&\implies c = 1001
\end{aligned}$$

(C $P(1 - k^{500} \leq \frac{1}{10000}|1000 \text{ successes})$

$$\begin{aligned}
P(1 - k^{500} \leq \frac{1}{10000}|1000 \text{ successes}) &= \int_{0.99999979999}^{1} 1001k^{1000}\, dk \\
&= k^{1001}|_{0.99999979999}^{1} \\
&= 1 - 0.999799810011 \\
&= 0.000200189989
\end{aligned}$$

Part 2 Updating by inspecting the posterior

(A) Number of successes for the posterior distribution $P(k|? \text{ successes}) = 5001k^{5000}$ also the likelihood function is $\mathcal{L}(k|? \text{ successes})$ so using this,

$$\begin{aligned}
\mathcal{L}(k|? \text{ successes}) &= k \cdot k \cdot ...? \text{ times} \cdot k \\
&= k^{?} \\
&= k^{5000} \\
&\implies ? = 5000
\end{aligned}$$

Therefore there are 5000 rolls.

(B) Normalized $\mathcal{L}(k|7000 \text{ successes})$

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathcal{L}(k|7000 \text{ successes})\, dk &= \int_{-\infty}^{\infty} c \cdot k \cdot k \cdot k ... 6996 \text{ times} ... \cdot k\, dk \\
&= \int_{0}^{1} c \cdot k^{7000}\, dk \\
&= c \cdot \frac{k^{7001}}{7001}|_{0}^{1} \\
&\implies c = 7001
\end{aligned}$$

Part 3 Bayesian updating

(A) Filling in the "2000 more successes" blanks

$$\text{prior distribution: } P(k) = 5001k^{5000}$$
$$\text{likelihood: } P(2000 \text{ more successes}|k) = 2001k^{2000}$$
$$\text{prior} \cdot \text{likelihood: } P(2000 \text{ more successes}|k) \cdot p(k) = 5001k^{5000} \cdot 2001k^{2000} = 10007001k^{7000}$$
$$\text{posterior distribution: } P(2000 \text{ more successes}|k) = \int_0^1 c \cdot 10007001k^{7000}\, dk \implies c \cdot \frac{10007001}{7001} = 1$$
$$\implies c = \frac{7001}{10007001} \implies P(2000 \text{ more successes}|k) = 7001k^{7000}$$

Part 4 Inference

(A) Smallest N such that $P(1 - k^{500} \leq \frac{1}{10000}|N \text{ successes}) = 0.99$

$$\begin{aligned}
P(1 - k^{500} \leq \frac{1}{10000}|N \text{ successes}) &= \int_{0.99999979999}^{1} (N+1)k^N\, dk \\
&= k^{N+1}|_{0.99999979999}^{1} \\
&= 1 - 0.99999979999^{N+1} \\
&\implies 1 - 0.99999979999^{N+1} = 0.99 \\
&\implies 0.99999979999^{N+1} = 0.01 \\
&\implies N + 1ln(0.99999979999) = ln(0.01) \\
&\implies N = \frac{ln(0.01)}{ln(0.99999979999)} - 1 \\
&\implies N = 23024696.3916
\end{aligned}$$

The amount of more successes after 5000 are $23024696.3916 - 5000 = 23019696.3916$