

Assignment 49-2

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1 Bayesian inference with a uniform prior

Start by assuming a uniform prior distribution (i.e. you know nothing about k until you've collected some data). So, k initially follows the uniform distribution over the interval $[0,1]$: $P(k) = \frac{1}{1-0} = 1, k \in [0, 1]$

1.a Other people make 1000 successful trips through the wormhole with no failures. What is the likelihood function for k given these 1000 successful trips?

The likelihood function for k is $L(k|1000 \text{ successes})$ which can be written as $[P(k)]^{1000} = k^{1000}$

1.b What is the posterior distribution for k given these 1000 successful trips? (This is the same as just normalizing the likelihood function). The posterior distribution for k will be ck for some c such that

$$\int_{-\infty}^{\infty} c \cdot L(k|1000 \text{ successes}) dk = 1$$

or

$$c \int_0^1 k^{1000} dk = c \left[\frac{k^{1001}}{1001} \right]_0^1 = \frac{c}{1001} = 1, c = 1001$$

So the posterior distribution is $1001k^{1000}$

1.c Assuming that you will use the wormhole 500 times per year, what is the posterior probability that the risk of disappearing forever into the wormhole is no more than your risk of dying from a car crash in a given year (1 in 10000) To find the probability that you will fail assuming that you travel 500 times a year that is the same as finding $P(1 - k^{500 \frac{1}{10000}} | 1000 \text{ successes})$

$$\int_{0.99999979999}^1 1001(k)^{1000} dk = (k)^{1001} \Big|_{0.99999979999}^1 = 1 - (0.99999979999)^{1001} = 0.0002$$

2 Updating by inspecting the posterior

You keep on collecting data until your posterior distribution is $P(k|?successes) = 5001k^{5000}$. But then you forget how many successes you have counted. Because this is a rather simple scenario, it's easy to find this number by inspecting your posterior distribution.

2.a Looking at the given posterior distribution, how many successes have you counted?

I've counted 5000

2.b Suppose you observe 2000 more successes. What is the posterior distribution now?

$7001k^{7000}$

3 Bayesian updating

To get some practice with the general procedure of Bayesian inference with a non-uniform prior, let's re-do Part 2, supposing you weren't able to remember the number of successes by inspecting your posterior distribution.

This time, you'll use $P(k|?successes) = 5001k^{5000}$ as your prior distribution.

3.a Suppose you observe 2000 more successes. Fill in the blanks:

Prior distribution: $P(k) = 5001k^{5000}$

Likelihood: $P(2000\text{ more successes}|k) = k^{2000}$

Prior \times Likelihood: $P(2000 \text{ more successes}|k)P(k) = k^{2000} \cdot 5001k^{5000}$

Posterior Distribution: $P(k|2000 \text{ more successes}) = 7001k^{7000}$

4 Inference

Let's go back to the moment when you forgot the number of successes, and your posterior distribution was $P(k | \text{? successes}) = 5001k5000$.

4.a Assuming that you will use the wormhole 500 times per year, how many more people do you need to observe successfully come out of the wormhole to be 99% sure the risk of disappearing forever into the wormhole is no more than your risk of dying from a car crash in a given year (1 in 10000)?

I need to evaluate $P(1k^{500} \frac{1}{10000} | N \text{ more successes})$. But instead of finding that probability given that $N = 1000$, we are now finding N such that the probability is 0.99.

$$\int_{0.99999979999}^1 (N+1)k^N dk = k^{N+1} \Big|_{0.99999979999}^1 = 1 - (0.99999979999)^{N+1}$$

Now we need to solve this equation for 0.99

$$1 - (0.99999979999)^{N+1} = 0.99 \rightarrow 0.01 = (0.99999979999)^{N+1} \rightarrow$$

$$\frac{0.01}{0.99999979999} = 0.99999979999^N \rightarrow 0.0100000002 = 0.99999979999^N$$

$$N = 23024699$$