Problem 50-1

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1 Given the data, compute the likelihood P(two rolls of 1 and three rolls of 2—switch). Your work should lead to the following result:

$$L(two\ rolls\ of\ 1\ and\ 3\ rolls\ of\ 2|switch) =$$

$$(5choose2) \cdot P(roll\ of\ 1) \cdot P(roll\ of\ 1) \cdot P(roll\ of\ 2) \cdot P(roll\ of\ 2) \cdot P(roll\ of\ 2) =$$

We find the probability based on a fair die and when it is based on a die with 3 sides of 1 and 3 sides of 2.

Fair Die:

$$L(two\ rolls\ of\ 1\ and\ 3\ rolls\ of\ 2|switch) = (5C2) \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} =$$

$$\frac{1}{6^5} = \frac{5!}{2! \cdot 3!} 0.0001286 = 0.001286$$

Biased Die(3 sides of 1, 3 sides of 2):

$$L(two\ rolls\ of\ 1\ and\ 3\ rolls\ of\ 2|switch) = (5C2) \cdot \frac{3}{6} \cdot \frac{3}{6} \cdot \frac{3}{6} \cdot \frac{3}{6} \cdot \frac{3}{6} = \frac{3^5}{6^5} =$$

$$\frac{5!}{2! \cdot 3!} 0.03125 = 0.3125$$

$$P(two\ rolls\ of\ 1\ and\ 3\ rolls\ of\ 2|switch) = \begin{cases} 0.3125 & switch = True \\ 0.001286 & switch = False \end{cases}$$

Suppose that, before the magician rolled the die, you were agnostic: you believed there was a 50/50 chance that the die was fair.

a. Given your prior belief, what is the prior distribution P(switch)?

$$P(switch) = \begin{cases} 0.5 & switch = True \\ 0.5 & switch = False \end{cases}$$

b. What is the posterior distribution P(switch|two rolls of 1 and three rolls of 2)?

We know that P(parameter|data) = $\propto$P(data|parameter)$\cdot$ P(parameter) by Bayes theorem. So P(data|parameter) = P(two rolls of 1 and three rolls of 2|switch) and P(parameter) = P(switch).

$P(switch|two\ rolls\ of\ 1\ and\ 3\ rolls\ of\ 2) \propto P(two\ rolls\ of\ 1\ and\ 3\ rolls\ of\ 2|switch) \cdot P(switch) =$

$$\begin{cases} 0.3125 & switch = True \\ 0.001286 & switch = False \end{cases} \cdot \begin{cases} 0.5 & switch = True \\ 0.5 & switch = False \end{cases} = \begin{cases} 0.15625 & switch = True \\ 0.000643 & switch = False \end{cases}$$

It has to sum up to 1 so we solve

$$c(0.15625 + 0.000643) = 1 \rightarrow c = \frac{1}{0.156893} = 6.37377$$

So:

$$P(switch|two\ rolls\ of\ 1\ and\ 3\ rolls\ of\ 2) = \begin{cases} 0.15625 \cdot 6.37377 = 0.996 & switch = True \\ 0.000643 \cdot 6.37377 = 0.004, & switch = False \end{cases}$$

Suppose that, before the magician rolled the die, you were optimistic: you believed there was a 99% chance that the die was fair.

a. Given your prior belief, what is the prior distribution P(switch)?

$$P(switch) = \begin{cases} 0.01 & switch = True \\ 0.99 & switch = False \end{cases}$$

b. What is the posterior distribution P(switch|two rolls of 1 and three rolls of 2)?

Like in Part 1 we will find P(switch—two rolls of 1 and three rolls of 2)

$P(switch|two\ rolls\ of\ 1\ and\ 3\ rolls\ of\ 2) \propto P(two\ rolls\ of\ 1\ and\ 3\ rolls\ of\ 2|switch) \cdot P(switch) =$

$$\begin{cases} 0.3125 & switch = True \\ 0.001286 & switch = False \end{cases} \cdot \begin{cases} 0.01 & switch = True \\ 0.99 & switch = False \end{cases} = \begin{cases} 0.003125 & switch = True \\ 0.00127314 & switch = False \end{cases}$$

It has to sum up to 1 so we solve

$$c(0.003125 + 0.00127314) = 1 \rightarrow c = \frac{1}{0.00439814} = 227.3688$$

So:

$$P(switch|two\ rolls\ of\ 1\ and\ 3\ rolls\ of\ 2) = \begin{cases} 227.3688 * 0.003125 = 0.710 & switch = True \\ 227.3688 * 0.00127314 = 0.289, & switch = False \end{cases}$$

Suppose that, before the magician rolled the die, you were pessimistic: you believed there was a 1% chance that the die was fair.

a. Given your prior belief, what is the prior distribution P(switch)?

$$P(switch) = \begin{cases} 0.99 & switch = True \\ 0.01 & switch = False \end{cases}$$

b. What is the posterior distribution P(switch—two rolls of 1 and three rolls of 2)?

Again we find P(switch—two rolls of 1 and three rolls of 2)

$P(switch|two\ rolls\ of\ 1\ and\ 3\ rolls\ of\ 2) \propto P(two\ rolls\ of\ 1\ and\ 3\ rolls\ of\ 2|switch) \cdot P(switch) =$

$$\begin{cases} 0.3125 & switch = True \\ 0.001286 & switch = False \end{cases} \cdot \begin{cases} 0.99 & switch = True \\ 0.01 & switch = False \end{cases} = \begin{cases} 0.309375 & switch = True \\ 0.00001286 & switch = False \end{cases}$$

It has to sum up to 1 so we solve

$$c(0.309375 + 0.00001286) = 1 \rightarrow c = \frac{1}{0.30938786} = 3.23$$

So:

$$P(switch|two\ rolls\ of\ 1\ and\ 3\ rolls\ of\ 2) = \begin{cases} 3.232188 \cdot 0.309375 = 0.99996, & switch = True \\ 3.232188 \cdot 0.00001286 = 0.00004, & switch = False \end{cases}$$