Machine Learning Assignment 52

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Problem 1

(a) *Using the identity* $Var[X] = E[X^2] - E[X]^2$*, compute* $Var[X]$ *if* X *is sampled from the continuous uniform distribution* $U[a, b]$

$$
\begin{aligned}
E[X] &= \int_a^b xp(x)\mathrm{d}x \\
&= \frac{1}{2(b-a)} \times x^2\Big|_a^b \\
&= \frac{a^2 - b^2}{2(b-a)} \\
&= \frac{a+b}{2} \\
E[X^2] &= \int_a^b x^2 p(x)\mathrm{d}x \\
&= \frac{1}{3(b-a)} \times x^3\Big|_a^b \\
&= \frac{b^3 - a^3}{3(b-a)} \\
&= \frac{a^2 + ab + b^2}{3} \\
Var[X] &= E[X^2] - E[X]^2 \\
&= \frac{a^2 + ab + b^2}{3} - \left(\frac{a+b}{2}\right)^2 \\
&= 4\frac{a^2 + ab + b^2}{12} - 3\frac{a^2 + 2ab + b^2}{12} \\
&= \frac{a^2 - 2ab + b^2}{12} \\
&= \frac{(b-a)^2}{12}
\end{aligned}
$$

(b) *Using the identity* $Var[X] = E[X^2] - E[X]^2$, *compute* $Var[X]$ *if* X *is sampled from the exponential distribution* $p(x) = \lambda e^{-\lambda x}, x \geq 0$.

$$\begin{aligned}
E[X] &= \int_0^\infty x * \lambda e^{-\lambda x} \mathrm{d}x \\
&= \left(e^{-\lambda x} * (x + \frac{1}{\lambda}) \right) \Big|_0^\infty \\
&= \frac{1}{\lambda} \\
E[X^2] &= \int_0^\infty x^2 * \lambda e^{-\lambda x} \mathrm{d}x \\
&= \left(x^2 e^{-\lambda x} - \frac{2xe^{-\lambda x}}{\lambda} - \frac{2e^{-\lambda x}}{\lambda^2} \right) \Big|_0^\infty \\
&= \frac{2}{\lambda^2} \\
Var[X] &= E[X^2] - E[X]^2 \\
&= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda} \right)^2 \\
&= \frac{1}{\lambda^2}
\end{aligned}$$

(c) *Using the identity* $Var[N] = E[N^2] - E[N]^2$, *compute* $Var[N]$ *if* N *is sampled from the Poisson distribution* $p(n) = \dfrac{\lambda^n e^{-\lambda}}{n!}, n \in \{0, 1, 2, \ldots\}$.

$$
\begin{aligned}
E[N] &= \sum_{0}^{\infty} \frac{n\lambda^n e^{-\lambda}}{n!} \\
&= e^{-\lambda} \sum_{1}^{\infty} \frac{\lambda^n}{(n-1)!} \\
&= \lambda e^{-\lambda} \sum_{1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} \\
&= \lambda e^{-\lambda} e^{\lambda} \\
&= \lambda \\
E[N^2] &= \sum_{0}^{\infty} \frac{n^2\lambda^n e^{-\lambda}}{n!} \\
&= e^{-\lambda} \sum_{1}^{\infty} \frac{n\lambda^n}{(n-1)!} \\
&= \lambda e^{-\lambda} \sum_{1}^{\infty} \frac{n\lambda^{(}n-1)}{(n-1)!} \\
&= \lambda e^{-\lambda} \sum_{0}^{\infty} \frac{(n+1)\lambda^n}{n!} \\
&= \lambda e^{-\lambda} \left(\sum_{0}^{\infty} \frac{n\lambda^n}{n!} + \sum_{1}^{\infty} \frac{\lambda^n}{n!} \right) \\
&= \lambda e^{-\lambda} \left(\sum_{1}^{\infty} \frac{\lambda\lambda^{(}n-1)}{(n-1)!} + e^{\lambda} \right) \\
&= \lambda e^{-\lambda} \left(\lambda \sum_{0}^{\infty} \frac{\lambda^n}{n!} + e^{\lambda} \right) \\
&= \lambda e^{-\lambda} \left(\lambda e^{\lambda} + e^{\lambda} \right) \\
&= \lambda(\lambda+1) \\
Var[N] &= E[N^2] - E[N]^2 \\
&= \lambda(\lambda+1) - \lambda^2 \\
&= \lambda
\end{aligned}
$$