Machine Learning Assignment 49

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Problem 1

(a) *Given that $X \sim p(x)$, where p(x) is a continuous distribution, prove that for any real number a we have E[a X]=a E[X]*

Solution

$$E[ax] = \int (a * p(x)) * xdx$$

$$a * \int x * p(x) = a * E[x]$$

(b) *Given that $X_1, X_2 \sim p(x)$, where p(x) is a continuous distribution, prove that $E[X_1 + X_2] = E[X_1] + E[X_2]$*

Solution

$$E[X_1 + X_2] = \int (p(X_1) + p(X_2)) * xdx$$

$$\int p(X_1) * x + p(X_2) * xdx = \int p(X_1) * x + \int p(X_2) * x$$

$$E[X_1] + E[X_2]$$

(c) *Given that X∼p(x) where p(x) is a continuous probability distribution, prove the identity Var[X]=E[X^2] - E[X]^2*

Solution

$$Var[X] = \int (x - E[x])^2 * p(x)dx$$

$$\int (x^2 - 2xE[X] + E[X]^2) * p(x)$$

$$\int x^2 * p(x)dx - 2E[x] \int x * p(x)dx + \int E[x]^2 * p(x)$$

$$E[x^2] - 2E[x]^2 + E[x]^2 = E[x^2] - E[x]^2$$

(d) *Use bisection search to estimate $\sqrt{5}$ to 4 decimal places by hand, showing your work at each step of the way.*

Solution

Function $x^2 - 5$ from [2, 3]
First Estimate: 2.5
$2.5^2 = 6.25 > 5$ Update Bounds : [2, 2.5]
Second Estimate: 2.25
$2.25^2 = 5.0625 > 5$ Update Bounds : [2, 2.25]
Third Estimate 2.125
$2.125^2 = 4.515 < 5$ Update Bounds : [2.125, 2.25]
Fourth Estimate: 2.1875 $2.1875^2 = 4.78515625 < 5$
Update Bonds: [2.1875, 2.25]
Fifth Estimate:
$2.21875^2 = 4.9228515625 < 5$
New Bounds: [2.21875, 2.25]
Sixth Estimate: 2.23438
$2.23438^2 = 4.9924539844 < 5$
New Bounds: [2.23438, 2.25]
Seventh Estimate: 2.24219
$2.24219^2 = 5.0274159961 > 5$
New Bounds: [2.23438, 2.24219]
Eight Estimate: 2.23829
$2.23829^2 = 5.0099421241 > 5$
New Bounds: [2.23438, 2.23829]
Ninth Estimate: 2.23634
$2.23634^2 = 5.0012165956 > 5$
New Bounds : [2.23438, 2.23634]
Tenth Estimate: 2.23536
$2.23536^2 = 4.9968343296 < 5$
New Bounds: [2.23536, 2.23634]
Eleventh Estimate: 2.23585
$2.23585^2 = 4.9990252225 < 5$
New Bounds: [2.23585, 2.23634]
Twelfth Estimate: 2.2361
$2.2361^2 > 5$
New Bounds: [2.23585, 2.2361]
13th Estimate: 2.23598
$2.23598^2 < 5$
New Bounds: [2.23598, 2.2361]
14th Estimate: 2.23604
$2.23604^2 < 5$
New Bounds: [2.23604, 2.2361]
15th Estimate: 2.23607
$2.23607^2 > 5$
New Bounds: [2.23604, 2.23607]

4th Decimal Place Estimate: 2.2360

(e) *Use "merge sort" to sort the list [4,8,7,7,4,2,3,1]. Do the problem by hand and show your work at each step of the way.*

Solution

Divide [4,8,7,7,4,2,3,1] Into [4,8,7,7] and [4,2,3,1]
Divide[4,8,7,7] Into [4,8],[7,7]
Divide [4,8] Into [4] and [8]
Combine back into [4,8]
Divide [7,7] into [7] and [7]
Combine back into [7,7]
Combine back into [4,7,7,8]
Divide [4,2,3,1] Into [4,2] and [3,1]
Divide [4,2] into [4] and [2]
Combine back into [2,4]
Divide [3,1] Into [3] and [1]
Combine back into [1,3]
Combine back into [1,2,3,4]
Combine back into [1,2,3,4,4,7,7,8]

Problem 2

(1.a) *Other people make 1000 successful trips through the wormhole with no failures. What is the likelihood function for k given these 1000 successful trips?*

Solution

$$k^{1000}$$

(1.b) *What is the posterior distribution for k given these 1000 successful trips? (This is the same as just normalizing the likelihood function)*

Solution

$$\int_0^1 (x * k^{1000}) = 1$$

$$\frac{x * 1^{1001}}{1001} = 1$$

$$x = 1001$$

$$1001 * k^{1000}$$

(1.c) *Assuming that you will use the wormhole 500 times per year, what is the posterior probability that the risk of disappearing forever into the wormhole is* *no more than your risk of dying from a car crash in a given year (1 in 10000)?*

Solution

$$1 - k^{500} \leq \frac{1}{10000}$$
$$k = 0.99999979999$$

$$\int_k^1 1001 * k^{1000} dk = 0.00020019$$

(2.a) *Looking at the given posterior distribution, how many successes have you counted*

Solution

$$5000$$

(2.b) *Suppose you observe 2000 more successes. What is the posterior distribution now?*

Solution

$$7001 * k^{7000}$$

(3.a) *Suppose you observe 2000 more successes. Fill in the blanks*

Solution

likelihood : k^{2000}
prior x likelihood: $5001 * k^{7000}$
posterior distribution: $\int_0^1 x * 5001 * k^{7000} dk = 1$

$$\frac{x * 5001}{7001} = 1$$
$$x = \frac{7001}{5001}$$
$$x = 1.399992$$
$$7001 * k^{7000}$$

(4.a) *Assuming that you will use the wormhole 500 times per year, how many more people do you need to observe successfully come out of the wormhole to be* 99% *sure the risk of disappearing forever into the wormhole is no more than your risk of dying from a car crash in a given year (1 in 10000)?*

Solution

Refer to k as 0.99999979999

$$\int_k^1 (n + 5001) * k^{n+5000} dk = 0.99$$

$$\frac{(n+5001)}{n+5001} * k^{n+5001}\Big|_k^1 = 0.99$$

$$1 - (k)^{n+5001} = 0.99$$

$$(n+5001) * log(k) = log(0.01)$$

$$n = \frac{log(0.01)}{log(k)} - 5001$$

$$n = 23024697.3916 - 5001$$

$$n = 23019696.3916$$

Note: Rounding differences, I'm only 3 off