Assignment 53-1

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Given that $X \sim U[0,1], computeCov[X,X^2].$

$$Cov[X_1, X_2] = E[(X_1 - \overline{X_1})(X2 - \overline{X_2}].$$

$$Cov[X, X^2] = E[(X - E[X])(X^2 - E[X^2])] = \int_0^1 (x - E[X])(x^2 - E[X^2]) \cdot \frac{1}{1-0} dx =$$

$$\int_0^1 (x^3 - xE[X^2] = x^2 E[X] + E[X]E[X^2])dx$$

As an intermediate step I am going to find E[X] and $E[X^2]$ and plug those in.

$$E[X] = \int_0^1 x dx = \left. \frac{x^2}{2} \right|_0^1 = \frac{1}{2}$$

$$E[X^2] = \int_0^1 x^2 dx = \left. \frac{x^3}{3} \right|_0^1 = \frac{1}{3}$$

$$\int_0^1 (x^3 - \frac{x}{3} = \frac{x^2}{2} + \frac{1}{6})dx = \left. \frac{x^4}{4} - \frac{x^2}{6} - \frac{x^3}{6} + \frac{x}{6} \right|_0^1 = \frac{1}{4} - \frac{1}{6} - \frac{1}{6} + \frac{1}{6} = \frac{1}{12}$$

b. Given that $X_1, X_2 \sim U[0,1]$, compute $Cov[X_1, X_2]$.

$$Cov[X_1, X_2] = E[(x_1 - E[X_1])(x_2 - E[X_2])] = \int_0^1 \int_0^1 (x_1 - E[X_1])(x_2 - E[X_2])dx_1 dx_2 =$$

$$\int_0^1 \int_0^1 x_1 x_2 - x_1 E[X_2] - x_2 E[X_1] + E[X_1]E[X_2] dx_1 dx_2 =$$

Now we are going to find $E[X_1], E[X_2]$

$$E[X_1] = \int_0^1 x_1 dx_1 = \frac{(x_1)^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$E[X_2] = \int_0^1 x_2 dx_2 = \frac{(x_2)^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\int_0^1 \int_0^1 x_1 x_2 - \frac{x_1}{2} - \frac{x_2}{2} + \frac{1}{4} dx_1 dx_2 = \frac{(x_1)^2 x_2}{2} - \frac{(x_1)^2}{4} - \frac{x_2 x_1}{2} + \frac{x_1}{4}\Big|_0^1 =$$

$$\int_0^1 \frac{x_2}{2} - \frac{x_2}{2} dx_2 = \int_0^1 0 dx_2 = 0$$

c. Prove that $Var[X_1 + X_2] = Var[X_1] + Var[X_2] + 2Cov[X_1, X_2]$

$$Var[X_1 + X_2] = \int_{-\infty}^{\infty} (x_1 + x_2 - E[X_1 + X_2])^2 \cdot p(x) dx =$$

For now we will just be working with $(x_1 + x_2 - E[X_1 + X_2])^2$ and reducing it to a helpful form

My first step will be to expand out

$$(x_1 + x_2 - E[X_1 + X_2])^2 =$$

$$(x_1)^2 + (x_2)^2 + E[X_1]^2 + E[X_2]^2 + 2x_1 x_2 - 2x_1 E[X_1] - 2x_1 E[X_2] - 2x_2 E[X_1] - 2x_2 E[X_2] + 2E[X_1]E[X_2] =$$

$$[(x_1)^2 - 2x_1 E[X_1] + E[X_1]^2] + [(x_2)^2 - 2x_2 E[X_2] + E[X_2]^2] + 2(x_1 x_2 - x_1 E[X_2] - x_2 E[X_1] + E[X_1]E[X_2]) =$$

$$(x_1 - E[X_1])^2 + (x_2 - E[X_2])^2 + 2[(x_1 - E[X+1])(x_2 - E[X_2])]$$

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Now to go back to the integral with this new format

$$\int_{-\infty}^{\infty} (x_1 - E[X_1])^2 + (x_2 - E[X_2])^2 + 2[(x_1 - E[X+1])(x_2 - E[X_2])] \cdot p(x)dx =$$

$$\int_{-\infty}^{\infty} (x_1 - E[X_1])^2 \cdot p(x)dx + \int_{-\infty}^{\infty} (x_2 - E[X_2])^2 \cdot p(x)dx+$$

$$2 \int_{-\infty}^{\infty} (x_1 - E[X+1])(x_2 - E[X_2]) \cdot p(x)dx =$$

$$Var[X_1] + Var[X_2] + 2Cov[X_1, X_2]$$

d. Prove that $Cov[X_1, X_2] = E[X_1X_2]E[X_1]E[X_2]$

$$Cov[X_1, X_2] = \int_{-\infty}^{\infty} (x_1 - E[X_1])(x_2 - E[X_2]) \cdot p(x)dx =$$

$$\int_{-\infty}^{\infty} [x_1x_2 - x_1E[X_2] - x_2E[X_1] + E[X_1]E[X_2]] \cdot p(x)dx =$$

Keep in mind the following for the proceeding steps

- $\int_{-\infty}^{\infty} p(x)dx = 1$
- $\int_{-\infty}^{\infty} x \cdot p(x)dx = E[X]$
- E[X] is a constant and can be pulled out so $\int_{-\infty}^{\infty} E[X] \cdot p(x)dx = E[X]$

$$\int_{-\infty}^{\infty} x_1x_2 \cdot p(x)dx + \int_{-\infty}^{\infty} E[X_1]E[X_2] \cdot p(x)dx - \int_{-\infty}^{\infty} x_1E[X_2] \cdot p(x)dx - \int_{-\infty}^{\infty} x_2E[X_1] \cdot p(x)dx =$$

$$E[X_1X_2] + E[X_1]E[X_2] - E[X_2] \int_{-\infty}^{\infty} x_1 \cdot p(x)dx - E[X_1] \int_{-\infty}^{\infty} x_2 \cdot p(x)dx =$$

$$E[X_1X_2] + E[X_1]E[X_2] - E[X_2]E[X_1] - E[X_1]E[X_2] = E[X_1X_2] - E[X_1]E[X_2]$$

$$Cov[X_1, X_2] = E[X_1X_2] - E[X_1]E[X_2]$$

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